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Rothberger and Rothberger-type star selection principles on hyperspaces

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In this paper we characterize the Rothberger property and the selection principles star-Rothberger and strongly star-Rothberger in the hyperspaces $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ and $\mathbb{CS}(X)$, endowed with the Vietoris topology. To characterize the corresponding principles type star, we introduce a couple of technical selection principles, which we have denoted by $\mathbf{S}_{\Pi_V}(\Pi_V(\Lambda), \Pi_V(\Lambda))$ and $\mathbf{S}^*_{\Pi_V}(\Pi_V(\Lambda), \Pi_V(\Lambda))$. Also, we give an equivalence of the selection principle $\mathbf{S}_1(\mathcal{D}, \mathcal{D})$ in the same hyperspaces, by using c_V-covers of a space.

1. Introduction and preliminaries

The study of selection principles was initiated by Borel [1], Menger [6], Hurewicz [3], Rothberger [9], and others. The works of Scheepers [10] initiated a systematic investigation on selection principles, which led to a great deal of research into selection principles and their applications. This theory is now connected with several fields of mathematics, for example, set theory, function spaces and hyperspaces. One of the lines of research generated by the selection principles is concerning to selection principles type star. In 1999,

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Kočinac [4] started an important study on star selection principles. Currently, many authors have worked with these concepts to obtain results in hyperspaces.

We recall that given a topological space X, $CL(X)$ denotes the set of all nonempty closed subsets of X. The set $CL(X)$ usually receives the name of *hyperspace of* X. The hyperspace theory started in the early 20th century with the works of D. Pompeiu (1905), F. Hausdorff (1914) and L. Vietoris [11]. Topologies on $CL(X)$ and on other families of subsets of a topological space X were studied in 1951 by E. Michael [7]. Moreover, relations between numerous properties of the space X and their hyperspaces have been studied.

Relationships between selections principles and hyperspaces have been investigated. For example, in [2] the authors use the notion of π-networks to characterize topological spaces whose hyperspace, endowed with the upper Fell topology, satisfies the Rothberger property. Later, in [5] Li generalizes this notion by defining the concept of π_V-network and he uses this notion to study the Rothberger and Menger properties with the Vietoris topology. Recently, in [8] Osipov studied selection properties of the bitopological hyperspace $(2^X, \mathbf{F}^+, \mathbf{Z}^+)$, where $\mathbf{F}^+$ is the upper Fell topology and $\mathbf{Z}^+$ is the so called $\mathbf{Z}^+$-topology.

In this paper, motivated by the work of Li [5], we make a slight modification to the notion of π_V-network, as defined by Li, restricting the collection of open subsets on which π_V-networks act. This new notion is a $\pi_V(\Lambda)$-network (see Definition 2.1). Then:

A) We characterize the Rothberger property in the hyperspace $CL(X)$ and some subspaces of this one, endowed with the Vietoris topology.

B) We introduce a couple of selection principles, which we have denoted by the symbols $\mathbf{S}_{\Pi_V}(\Pi_V(\Lambda), \Pi_V(\Lambda))$ and $\mathbf{S}^*_{\Pi_V}(\Pi_V(\Lambda), \Pi_V(\Lambda))$ (see Definitions 2.7 and 2.10, respectively) to characterize the properties star-Rothberger and strongly star-Rothberger in $CL(X)$ and in some subspaces of this one, also endowed with the Vietoris topology.

Characterizations in *A*) and *B*) are obtained with the generic results in Theorems 2.5, 2.8 and 2.11.

We close the paper, again following ideas by Li, with a generalization of [5, Theorem 3.6] concerning c_V-covers of a space.

Next, we recall some notations and definitions. All spaces are assumed to be Hausdorff noncompact and, even, nonparacompact. Throughout this paper, ω denotes the first infinite cardinal. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of an infinite set X. The symbol $\mathbf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

- For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(b_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $b_n \in A_n$ and $(b_n : n \in \mathbb{N})$ is an element of $\mathcal{B}$.

This selection principle was defined in 1996 by M. Scheepers [10]. When $\mathcal{A}$ and $\mathcal{B}$ are both the collection $\mathcal{O}$ of open covers of a space X, then $\mathbf{S}_1(\mathcal{O}, \mathcal{O})$ defines the classical *Rothberger property* (see [9]).

For a set $A \subseteq X$ and a collection $\mathcal{U}$ of subsets of X, the star of A with respect to $\mathcal{U}$ is denoted by $St(A, \mathcal{U})$ and defined as the set $\bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. We write $St(x, \mathcal{U})$ instead of $St(\{x\}, \mathcal{U})$, if $A = \{x\}$ for some $x \in X$.

In [4] Kočinac defined the following principles. Both concepts are the star versions of the Rothberger property. A space X is:

- *Star-Rothberger* (SR) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{St(U_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of X.
- *Strongly star-Rothberger* (SSR) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there exists a sequence $(x_n : n \in \mathbb{N})$ of elements of X such that $\{St(x_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of X.

In the same paper, Kočinac [4, Theorem 2.9] showed that if X is a paracompact Hausdorff space, then the three Rothberger-type properties (SR, SSR and the Rothberger property) are equivalent.

On the other hand, given a space X, the hyperspace $CL(X)$ is considered with the Vietoris topology, which has as base the family:

$$\{\langle U_1, \ldots, U_n\rangle : n \in \mathbb{N} \text{ and } U_1, \ldots, U_n \text{ are open subsets of X}\},$$

where $\langle U_1, \ldots, U_n\rangle$ denotes the following subset of $CL(X)$:

$$\left\{A \in CL(X) : A \subseteq \bigcup_{i=1}^{n} U_i,\ A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, n\}\right\}.$$

Recall that another hyperspaces of a space X are $\mathbb{K}(X)$ and $\mathbb{F}(X)$ which denote the family of all nonempty compact subsets and all nonempty finite subsets of X, respectively. By $\mathbb{CS}(X)$ we denote the family of all convergent sequences of X. For these hyperspaces, in this paper we will consider only the Vietoris topology, although in the literature there are another topologies that can be defined on these hyperspaces.

2. Rothberger and Rothberger-type star selection principles

For a subset $U \subseteq X$ and a family $\mathcal{U}$ of subsets of X, we write:

$$\begin{aligned} U^c &= X \backslash U; \\ \mathcal{U}^c &= \{U^c : U \in \mathcal{U}\}. \end{aligned}$$

Recall, too, that if A is a set, $[A]^{<\omega}$ denotes the set of all finite subsets of A.

Following the notation of [5], ζ denote the family:

$$\zeta = \{(V_1, \ldots, V_n) : V_1, \ldots, V_n \text{ are open subsets of } X, \quad n \in \mathbb{N}\}.$$

In [5], Li defined the concept of π_V-networks in a space X as follows:
A family ζ is called a *π_V-network of X* if for each open subset U of X, with $U \neq X$, there exist $(V_1, \ldots, V_n) \in \zeta$ and a finite set F with $F \cap V_i \neq \emptyset$ $(1 \leq i \leq n)$ such that $\bigcap_{i=1}^{n} V_i^c \subseteq U$ and $F \cap U = \emptyset$. The collection of π_V-networks of a space X is denoted by Π_V.

Note that in the previous definition, π_V-networks of X act on all open subsets of the space. In this paper, instead of all open subsets of X, we will only consider the elements in Λ^c, where $\Lambda \subseteq CL(X)$ satisfies:

(a) If $A \in \Lambda$, then $\{x\} \in \Lambda$, for each $x \in A$;
(b) Λ is closed under finite unions.

We refer to these types of networks as $\pi_V(\Lambda)$-network of X. Namely:

Definition 2.1. A family ζ is called a $\pi_V(\Lambda)$-network of X, if for each $U \in \Lambda^c$, there exist a $(V_1, \ldots, V_n) \in \zeta$ and $F \in [X]^{<\omega}$ with $F \cap V_i \neq \emptyset$ $(1 \leq i \leq n)$ such that $\bigcap_{i=1}^{n} V_i^c \subset U$ and $F \cap U = \emptyset$.

We write $\Pi_V(\Lambda)$ to denote the collection of $\pi_V(\Lambda)$-networks of X. Note that if $\Lambda = CL(X)$, we have that $\Pi_V(\Lambda)$ is Π_V as defined by Li. In the following example, we present a family Λ which induces a $\pi_V(\Lambda)$-network such that is not a π_V-network.

Example 2.2. Let $X = (0,1) \cup (3,\infty)$ be a space with the relative topology of $\mathbb{R}$ and put $Q = \mathbb{Q} \cap (3,\infty)$. Let Λ be the collection of all $A \subseteq X$ such that A is compact in X and $A \subseteq (0,1) \cup Q$. For each $x \in X$, we denote $\mathcal{V}_x = \{(x - \frac{1}{n}, x + \frac{1}{n}) \cap X : n \in \mathbb{N}\}$ and $\mathcal{V} = \bigcup\{\mathcal{V}_x : x \in X\}$. Define $\zeta = \{(V_1,\dots,V_m) : V_1,\dots,V_m \in \mathcal{V}$ and $m \in \mathbb{N}\}$. Then, we have the following results.

Fact 1. ζ is a $\pi_V(\Lambda)$-network of X. Indeed, take any $U \in \Lambda^c$. There exist $A \subseteq (0,1)$ and $Q' \in [Q]^{<\omega}$ such that A is a compact subset of X and $U^c = A \cup \{x : x \in Q'\}$. Since A is compact, there exist $V_1,\dots,V_n \in \mathcal{V}$ such that $A \subseteq \bigcup_{i=1}^n V_i$ and for each $i \in \{1,\dots,n\}$, $A \cap V_i \neq \emptyset$. Moreover, since Q' is a finite subset, we can renumber Q' as $Q' = \{x_{n+1},\dots,x_m\}$. For each $x_j \in Q'$, $j \in \{n+1,\dots,m\}$, we fix $V_j \in \mathcal{V}$ such that $x_j \in V_j$ and $A \cap V_j = \emptyset$. It is not difficult to prove that $U^c \in \langle V_1,\dots,V_m\rangle$. Observe that $(V_1,\dots,V_m) \in \zeta$. On the other hand, for each $i \in \{1,\dots,n\}$, we take $x_i \in A \cap V_i$ and we put $F = \{x_i : i \in \{1,\dots,n\}\} \cup Q'$. Clearly $F \in [X]^{<\omega}$, $F \cap U = \emptyset$ and for each $i \in \{1,\dots,m\}$, $F \cap V_i \neq \emptyset$. Moreover, since $U^c \subseteq \bigcup_{i=1}^m V_i$, we have that $\bigcap_{i=1}^m V_i^c \subseteq U$. Therefore, ζ is a $\pi_V(\Lambda)$-network of X.

Fact 2. ζ is not a π_V-network of X. To see it, let $U = (0,1)$. It is clear that $U^c \notin \Lambda$. Suppose that $\bigcap_{i=1}^m V_i^c \subseteq U$, for some $(V_1,\dots,V_m) \in \zeta$. Hence, $(3,\infty) \subseteq \bigcup_{i=1}^m V_i$, which is not possible.

Lemma 2.3. *If $\zeta = \{(V_1,\dots,V_m) : V_1,\dots,V_m$ are open subsets of $X, \quad m \in \mathbb{N}\}$ is a $\pi_V(\Lambda)$-network of X, then the family:*

$$\mathcal{U} = \{\langle V_1,\dots,V_m\rangle : (V_1,\dots,V_m) \in \zeta\}$$

is an open cover of Λ.

Proof. Let $A \in \Lambda$. We have $A^c \in \Lambda^c$. Given that ζ is a $\pi_V(\Lambda)$-network, there exist $(V_1,\dots,V_m) \in \zeta$ and $F \in [X]^{<\omega}$ with $F \cap V_i \neq \emptyset$ $(1 \le i \le m)$ such that $\bigcap_{i=1}^m V_i^c \subseteq A^c$ and $F \cap A^c = \emptyset$. Note that $A \subseteq \bigcup_{i=1}^m V_i$. Moreover, since $F \subseteq A$ and $F \cap V_i \neq \emptyset$ $(1 \le i \le m)$, then for each $i \in \{1,\dots,m\}$, $A \cap V_i \neq \emptyset$. So, we obtain that $A \in \langle V_1,\dots,V_m\rangle$. Hence, $A \in \bigcup \mathcal{U}$. Therefore, the collection $\mathcal{U}$ is an open cover of Λ. □

Lemma 2.4. *If $\mathcal{U} = \{\langle V_1,\dots,V_m\rangle : V_1,\dots,V_m$ are open subsets of $X,\ m \in \mathbb{N}\}$ is an open cover of Λ, then the family:*

$$\zeta = \{(V_1,\dots,V_m) : \langle V_1,\dots,V_m\rangle \in \mathcal{U}\}$$

is a $\pi_V(\Lambda)$-network of X.

Proof. Pick any $U \in \Lambda^c$. Since $U^c \in \Lambda$ and $\mathcal{U}$ covers Λ, there are open subsets $V_1,\dots,V_m$ of X such that $U^c \in \langle V_1,\dots,V_m\rangle$. Note that $(V_1,\dots,V_m) \in \zeta$. Since for each $i \in \{1,\dots,m\}$, $U^c \cap V_i \neq \emptyset$, we can take $x_i \in U^c \cap V_i$. Put $F = \{x_i : i \in \{1,\dots,m\}\}$. We have that $F \in [X]^{<\omega}$ and for each $i \in \{1,\dots,m\}$, $F \cap V_i \neq \emptyset$. Moreover, given that $U^c \subseteq \bigcup_{i=1}^m V_i$, then $\bigcap_{i=1}^m V_i^c \subseteq U$. Finally, since $x_i \in U^c$, then $F \cap U = \emptyset$. Therefore, ζ is a $\pi_V(\Lambda)$-network of X. □

In the next theorem, we apply the previous lemmas to obtain a generic characterization of the Rothberger property in hyperspaces.

Theorem 2.5. *Given a topological space X, the following conditions are equivalent:*

(1) *Λ has the Rothberger property;*
(2) *X satisfies $\mathbf{S}_1(\Pi_V(\Lambda), \Pi_V(\Lambda))$.*

Proof. (1) ⇒ (2). Let $(J_n : n \in \mathbb{N})$ be a sequence in $\Pi_V(\Lambda)$. We put, for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{\langle V_1, \ldots, V_m\rangle : (V_1, \ldots, V_m) \in J_n\}$. From Lemma 2.3, we have that for each $n \in \mathbb{N}$, $\mathcal{U}_n$ is an open cover of Λ.

Then, applying (1) to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$, we can choose, for each $n \in \mathbb{N}$, $\langle V_1^n, \ldots, V_{m_n}^n\rangle \in \mathcal{U}_n$ such that the collection $\{\langle V_1^n, \ldots, V_{m_n}^n\rangle : n \in \mathbb{N}\}$ is an open cover of Λ.

Now, we prove that the collection $\mathcal{J} = \{(V_1^n, \ldots, V_{m_n}^n) : n \in \mathbb{N}\}$ is a $\pi_V(\Lambda)$-network of X. Indeed, let $U \in \Lambda^c$. So, $U^c \in \Lambda$. Since the collection $\{\langle V_1^n, \ldots, V_{m_n}^n\rangle : n \in \mathbb{N}\}$ is an open cover of Λ, there exists $k \in \mathbb{N}$ such that $U^c \in \langle V_1^k, \ldots, V_{m_k}^k\rangle$. Note that $(V_1^k, \ldots, V_{m_k}^k) \in \mathcal{J}$. On the other hand, since for each $i \in \{1, \ldots, m_k\}$, $U^c \cap V_i^k \neq \emptyset$, we can take $x_i \in U^c \cap V_i^k$. Define $F = \{x_i : i \in \{1, \ldots, m_k\}\}$. We have that $F \in [X]^{<\omega}$ and, for each $i \in \{1, \ldots, m_k\}$, $F \cap V_i^k \neq \emptyset$. Moreover, given that $U^c \subseteq \bigcup_{i=1}^{m_k} V_i^k$, then $\bigcap_{i=1}^{m_k} (V_i^k)^c \subseteq U$. Finally, since $F \subseteq U^c$, $F \cap U = \emptyset$. Therefore, $\mathcal{J}$ is a $\pi_V(\Lambda)$-network of X.

(2) ⇒ (1). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Λ. Without loss of generality, we can suppose that, for each $n \in \mathbb{N}$, the open cover $\mathcal{U}_n$ consists of basic open sets in $CL(X)$, that is, $\mathcal{U}_n = \{\langle V_{1,s}^n, \ldots, V_{m_s,s}^n\rangle : s \in S_n\}$, where $V_{i,s}^n$ is an open subset of X, for every $i \in \{1, \ldots, m_s\}$.

For each $n \in \mathbb{N}$, let $J_n = \{(V_{1,s}^n, \ldots, V_{m_s,s}^n) : s \in S_n\}$. From Lemma 2.4, we have that for each $n \in \mathbb{N}$, J_n is a $\pi_V(\Lambda)$-network of X.

If we apply (2) to the sequence $(J_n : n \in \mathbb{N})$, it is possible to choose, for each $n \in \mathbb{N}$, an element $(V_{1,s_n}, \ldots, V_{m_{s_n},s_n}) \in J_n$ such that the collection $\{(V_{1,s_n}, \ldots, V_{m_{s_n},s_n}) : n \in \mathbb{N}\}$ is a $\pi_V(\Lambda)$-network of X. Next, we prove that the family $\{\langle V_{1,s_n}, \ldots, V_{m_{s_n},s_n}\rangle : n \in \mathbb{N}\}$ is an open cover of Λ. For that, take any $A \in \Lambda$. So $A^c \in \Lambda^c$. Given that $\{(V_{1,s_n}, \ldots, V_{m_{s_n},s_n}) : n \in \mathbb{N}\}$ is a $\pi_V(\Lambda)$-network of X, there exists $F \in [X]^{<\omega}$ with $F \cap V_{i,s_n} \neq \emptyset$ $(1 \leq i \leq m_{s_n})$ such that $\bigcap_{i=1}^{m_{s_n}} (V_{i,s_n})^c \subseteq A^c$ and $F \cap A^c = \emptyset$. Note that $A \subseteq \bigcup_{i=1}^{m_{s_n}} V_{i,s_n}$. Moreover, since $F \subseteq A$ and $F \cap V_{1,s_n} \neq \emptyset$ $(1 \leq i \leq m_{s_n})$, we obtain that for each $i \in \{1, \ldots, m_{s_n}\}$, $A \cap V_{1,s_n} \neq \emptyset$. Hence, $A \in \langle V_{1,s_n}, \ldots, V_{m_{s_n},s_n}\rangle$. Thus $\{\langle V_{1,s_n}, \ldots, V_{m_{s_n},s_n}\rangle : n \in \mathbb{N}\}$ is an open cover for Λ. Therefore, Λ has the Rothberger property. □

Corollary 2.6. *Let X be a topological space and let Λ be one of the following hyperspaces: $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$. Then Λ has the Rothberger property if and only if X satisfies $\mathbf{S}_1(\Pi_V(\Lambda), \Pi_V(\Lambda))$.*

Note that if $\Lambda = CL(X)$ in Corollary 2.6, we obtain [5, Theorem 3.12].

Below, we characterize SSR property for some hyperspaces of $CL(X)$ such as $\mathbb{K}(X)$, $\mathbb{F}(X)$ and $\mathbb{CS}(X)$. With this aim, we introduce the following selection principle related with $\pi_V(\Lambda)$-networks of X.

From now on, if J_n is an element in $\Pi_V(\Lambda)$, we put:

$$J_n = \{(V_{1,s}^n, \ldots, V_{m_s,s}^n) : s \in S_n\}.$$

Definition 2.7. Let X be a topological space. We define:
$\mathbf{S}_{\Pi_V}(\Pi_V(\Lambda), \Pi_V(\Lambda))$: For each sequence $(J_n : n \in \mathbb{N}) \subseteq \Pi_V(\Lambda)$, there exists a sequence $(U_n : n \in \mathbb{N})$ in Λ^c such that

$$\mathcal{J} = \bigcup_{n\in\mathbb{N}} \left\{ (V_{1,s}^n, \ldots, V_{m_s,s}^n) \in J_n : \bigcap_{i=1}^{m_s} (V_{i,s}^n)^c \subseteq U_n, V_{i,s}^n \nsubseteq U_n \ (1 \leq i \leq m_s) \right\}$$

is an element of $\Pi_V(\Lambda)$.

Theorem 2.8. *Given a topological space X, the following conditions are equivalent:*

(1) *Λ is SSR;*
(2) *X satisfies $\mathbf{S}_{\Pi_V}(\Pi_V(\Lambda), \Pi_V(\Lambda))$.*

Proof. (1) $\Rightarrow$ (2): Let $(J_n : n \in \mathbb{N})$ be a sequence of $\pi_V(\Lambda)$-networks of X. We put, for each $n \in \mathbb{N}$, $J_n = \{(V^n_{1,s}, \dots, V^n_{m_s,s}) : s \in S_n\}$. It follows from Lemma 2.3 that the collection $\mathcal{U}_n = \{\langle V^n_{1,s}, \dots, V^n_{m_s,s}\rangle : s \in S_n\}$ is an open cover of Λ, for each $n \in \mathbb{N}$.

Now, applying (1) to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$, we can choose $A_n \in \Lambda$, for each $n \in \mathbb{N}$, such that the collection $\{St(\{A_n\},\mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of Λ. We put, for each $n \in \mathbb{N}$, $U_n = A^c_n$. Hence, the sequence $(U_n : n \in \mathbb{N})$ is in Λ^c.

Let us show that the collection:

$$\mathcal{J} = \bigcup_{n\in\mathbb{N}} \left\{ (V^n_{1,s}, \dots, V^n_{m_s,s}) \in J_n : \bigcap_{i=1}^{m_s} (V^n_{i,s})^c \subseteq U_n, V^n_{i,s} \nsubseteq U_n \ (1 \le i \le m_s) \right\}$$

is a $\pi_V(\Lambda)$-network of X. For this end, let $U \in \Lambda^c$. Then $U^c \in \Lambda$ and therefore, there exists $n_0 \in \mathbb{N}$ such that $U^c \in St(\{A_{n_0}\},\mathcal{U}_{n_0})$. Then, there exists $\langle V^{n_0}_{1,s_0}, \dots, V^{n_0}_{m_{s_0},s_0}\rangle \in \mathcal{U}_{n_0}$ so that $\{U^c, A_{n_0}\} \subseteq \langle V^{n_0}_{1,s_0}, \dots, V^{n_0}_{m_{s_0},s_0}\rangle$. It follows that $\bigcap_{i=1}^{m_{s_0}} (V^{n_0}_{i,s_0})^c \subseteq U_{n_0}$ and for each $i \in \{1, \dots, m_{s_0}\}$, $V^{n_0}_{i,s_0} \nsubseteq U_{n_0}$. Thus, $(V^{n_0}_{1,s_0}, \dots, V^{n_0}_{m_{s_0},s_0}) \in \mathcal{J}$. Further, for each $i \in \{1, \dots, m_{s_0}\}$, we take $x_i \in U^c \cap V^{n_0}_{i,s_0}$. We put $F = \{x_i : i \in \{1, \dots, m_{s_0}\}\}$. Hence, $F \in [X]^{<\omega}$ with $F \cap V^{n_0}_{i,s_0} \neq \emptyset$. Moreover, since $U^c \subseteq \bigcup_{i=1}^{m_{s_0}} V^{n_0}_{i,s_0}$, then we obtain that $\bigcap_{i=1}^{m_{s_0}} (V^n_{i,s_0})^c \subseteq U$. Finally, $F \cap U = \emptyset$. We conclude that $\mathcal{J} \in \Pi_V(\Lambda)$.

(2) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Λ. Without loss of generality, we can suppose that, for each $n \in \mathbb{N}$, the open cover $\mathcal{U}_n$ consists of basic open subsets in $CL(X)$, that is $\mathcal{U}_n = \{\langle V^n_{1,s}, \dots, V^n_{m_s,s}\rangle : s \in S_n\}$, where $V^n_{i,s}$ is an open subset of X, for every $i \in \{1, \dots, m_s\}$. For each $n \in \mathbb{N}$, let $J_n = \{(V^n_{1,s}, \dots, V^n_{m_s,s}) : s \in S_n\}$. From Lemma 2.4, we obtain that each J_n is a $\pi_V(\Lambda)$-network of X.

We apply the condition (2) to the sequence $(J_n : n \in \mathbb{N})$ in $\Pi_V(\Lambda)$ to obtain a sequence $(U_n : n \in \mathbb{N})$ in Λ^c such that the collection:

$$\mathcal{J} = \bigcup_{n\in\mathbb{N}} \left\{ (V^n_{1,s}, \dots, V^n_{m_s,s}) \in J_n : \bigcap_{i=1}^{m_s} (V^n_{i,s})^c \subseteq U_n,\ V^n_{i,s} \nsubseteq U_n \ (1 \le i \le m_s) \right\}$$

is a $\pi_V(\Lambda)$-network of X. For each $n \in \mathbb{N}$, put $A_n = U^c_n$. Thus, the sequence $(A_n : n \in \mathbb{N})$ is in Λ.

Let us show that the collection $\{St(\{A_n\},\mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of Λ. Take any $A \in \Lambda$. Since $\mathcal{J}$ is a $\pi_V(\Lambda)$-network of X and $A^c \in \Lambda^c$, there exist $(V^{n_0}_{1,s_0}, \dots, V^{n_0}_{m_{s_0},s_0}) \in \mathcal{J}$ (for some $n_0 \in \mathbb{N}$, $s_0 \in S_{n_0}$) and a finite set $F \subseteq X$ such that for every $i \in \{1, \dots, m_{s_0}\}$, $F \cap V^{n_0}_{i,s_0} \neq \emptyset$, $\bigcap_{i=1}^{m_{s_0}} (V^{n_0}_{i,s_0})^c \subseteq A^c$ and $F \cap A^c = \emptyset$. Observe that, $\bigcap_{i=1}^{m_{s_0}} (V^{n_0}_{i,s_0})^c \subseteq U_{n_0}$ and for each $i \in \{1, \dots, m_{s_0}\}$, $V^{n_0}_{i,s_0} \nsubseteq U_{n_0}$. It means that $A_{n_0} \subseteq \bigcup_{i=1}^{m_{s_0}} V^{n_0}_{i,s_0}$ and, for each $i \in \{1, \dots, m_{s_0}\}$, $A_{n_0} \cap V^{n_0}_{i,s_0} \neq \emptyset$. Hence, $A_{n_0} \in \langle V^{n_0}_{1,s_0}, \dots, V^{n_0}_{m_{s_0},s_0}\rangle$. On the other hand, since $A \subseteq \bigcup_{i=1}^{m_{s_0}} V^{n_0}_{i,s_0}$ and $F \cap A^c = \emptyset$, it follows that $A \cap V^{n_0}_{i,s_0} \neq \emptyset$, for each $i \in \{1, \dots, m_{s_0}\}$. Thus, $A \in \langle V^{n_0}_{1,s_0}, \dots, V^{n_0}_{m_{s_0},s_0}\rangle$. We conclude that $A \in St(\{A_{n_0}\},\mathcal{U}_{n_0})$. This shows that the collection $\{St(\{A_n\},\mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of Λ. $\square$

We obtain the following particular cases by taking different choices of our family Λ.

Corollary 2.9. *Let X be a topological space and let Λ be one of the following hyperspaces: $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$. Then Λ is SRR if and only if X satisfies $\mathbf{S}_{\Pi_V}(\Pi_V(\Lambda), \Pi_V(\Lambda))$.*

Now, we give a characterization of the star-Rothberger property. Before of that, we introduce another useful principle related with $\pi_V(\Lambda)$-networks of X.

Definition 2.10. Let X be a topological space. We define
$\mathbf{S}^*_{\Pi_V}(\Pi_V(\Lambda), \Pi_V(\Lambda))$: For each sequence $(J_n : n \in \mathbb{N})$ in $\Pi_V(\Lambda)$, there exists $s(n) \in S_n$, $n \in \mathbb{N}$, such that

$$\mathcal{J} = \bigcup_{n\in\mathbb{N}} \Big\{(V^n_{1,s},\dots,V^n_{m_s,s}) \in J_n : \text{there exists } U \in \Lambda^c \text{ such that } \Big[\bigcap_{i=1}^{m_s}(V^n_{i,s})^c\Big] \cup \Big[\bigcap_{i=1}^{m_{s(n)}}(V^n_{i,s(n)})^c\Big] \subseteq U,\ V^n_{i,s} \nsubseteq U\ (1\le i\le m_s) \text{ and } V^n_{j,s(n)} \nsubseteq U\ (1 \le j \le m_{s(n)})\Big\}$$

is an element of $\Pi_V(\Lambda)$.

Theorem 2.11. *Given a topological space X, the following conditions are equivalent:*

(1) Λ *is SR;*
(2) X *satisfies* $\mathbf{S}^*_{\Pi_V}(\Pi_V(\Lambda),\Pi_V(\Lambda))$.

Proof. (1) $\Rightarrow$ (2): Let $(J_n : n\in\mathbb{N})$ be a sequence of $\pi_V(\Lambda)$-networks of X. We put, for each $n \in \mathbb{N}$, $J_n = \{(V^n_{1,s},\dots,V^n_{m_s,s}) : s\in S_n\}$. From Lemma 2.3, for each $n\in\mathbb{N}$, the collection $\mathcal{U}_n = \{\langle V^n_{1,s},\dots,V^n_{m_s,s}\rangle : s\in S_n\}$ is an open cover of Λ.

Now, applying (1) to the sequence $(\mathcal{U}_n : n\in\mathbb{N})$, we can choose for each $n\in\mathbb{N}$, $\langle V^n_{1,s(n)},\dots,V^n_{m_{s(n)},s(n)}\rangle \in \mathcal{U}_n$ such that the collection:

$$\{St(\langle V^n_{1,s(n)},\dots,V^n_{m_{s(n)},s(n)}\rangle,\mathcal{U}_n) : n\in\mathbb{N}\}$$

is an open cover of Λ.

We prove that the collection $\mathcal{J} = \bigcup_{n\in\mathbb{N}}\{(V^n_{1,s},\dots,V^n_{m_s,s}) \in J_n$: there exists $U\in\Lambda^c$ such that $[\bigcap_{i=1}^{m_s}(V^n_{i,s})^c] \cup [\bigcap_{i=1}^{m_{s(n)}}(V^n_{i,s(n)})^c] \subseteq U,\ V^n_{i,s} \nsubseteq U\ (1\le i\le m_s)$ and $V^n_{j,s(n)} \nsubseteq U\ (1\le j\le m_{s(n)}\}$ is a $\pi_V(\Lambda)$-network of X.

For this end, let $W\in\Lambda^c$. Then $W^c\in\Lambda$ and therefore, there exists $n_0\in\mathbb{N}$ such that $W^c \in St(\langle V^{n_0}_{1,s(n_0)},\dots,V^{n_0}_{m_{s(n_0)},s(n_0)}\rangle,\mathcal{U}_{n_0})$. Thus, there exists $\langle V^{n_0}_{1,s},\dots,V^{n_0}_{m_s,s}\rangle \in \mathcal{U}_{n_0}$ such that:

(i) $W^c \in \langle V^{n_0}_{1,s},\dots,V^{n_0}_{m_s,s}\rangle \in \mathcal{U}_{n_0}$;
(ii) $\langle V^{n_0}_{1,s},\dots,V^{n_0}_{m_s,s}\rangle \cap \langle V^{n_0}_{1,s(n_0)},\dots,V^{n_0}_{m_{s(n_0)},s(n_0)}\rangle \neq \emptyset$.

By (ii), we can consider $E \in \langle V^{n_0}_{1,s},\dots,V^{n_0}_{m_s,s}\rangle \cap \langle V^{n_0}_{1,s(n_0)},\dots,V^{n_0}_{m_{s(n_0)},s(n_0)}\rangle$. Put $U = E^c$. So $U\in\Lambda^c$. Moreover, $[\bigcap_{i=1}^{m_s}(V^{n_0}_{i,s})^c]\cup[\bigcap_{i=1}^{m_{s(n_0)}}(V^{n_0}_{i,s(n_0)})^c] \subseteq U$, $V^{n_0}_{i,s} \nsubseteq U(1\le i\le m_s)$ and $V^{n_0}_{j,s(n_0)} \nsubseteq U(1\le j\le m_{s(n_0)})$. Hence, we obtain that $(V^{n_0}_{1,s},\dots,V^{n_0}_{m_s,s}) \in \mathcal{J}$.

On the other hand, by (i), for each $i\in\{1,\dots,m_s\}$, there exists $x_i \in W^c\cap V^{n_0}_{i,s}$. Let $F = \{x_i : i\in\{1,\dots,m_s\}\}$. Since $F\subseteq W^c$, we have that $F\cap W = \emptyset$. Observe that, since for each $i\in\{1,\dots,m_s\}$, $F\subseteq V^{n_0}_{i,s}$, then $F\cap V^{n_0}_{i,s}\neq\emptyset$. Again, by (i), we obtain that $\bigcap_{i=1}^{m_s}(V^{n_0}_{i,s})^c \subseteq W$. Therefore, $\mathcal{J}\in\Pi_V(\Lambda)$.

(2) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n\in\mathbb{N})$ be a sequence of open covers of Λ. Without loss of generality, we can suppose that, for each $n\in\mathbb{N}$, the open cover $\mathcal{U}_n$ consists of basic open subsets in $CL(X)$, that is to say, $\mathcal{U}_n = \{\langle V^n_{1,s},\dots,V^n_{m_s,s}\rangle : s\in S_n\}$. From Lemma 2.4, it follows that each $J_n = \{(V^n_{1,s},\dots,V^n_{m_s,s}) : \langle V^n_{1,s},\dots,V^n_{m_s,s}\rangle \in\mathcal{U}_n\}$ is a $\pi_V(\Lambda)$-network of X.

We apply the condition (2) to the sequence $(J_n : n\in\mathbb{N})$ in $\Pi_V(\Lambda)$ to obtain $s(n)\in S_n$, $n\in\mathbb{N}$, such that: $\mathcal{J} = \bigcup_{n\in\mathbb{N}}\{(V^n_{1,s},\dots,V^n_{m_s,s}) \in J_n$: there exists $U\in\Lambda^c$ such that $[\bigcap_{i=1}^{m_s} V^c_{i,s}] \cup [\bigcap_{i=1}^{m_{s(n)}} V^c_{i,s(n)}] \subseteq U,\ V^n_{i,s} \nsubseteq U\ (1\le i\le m_s)$ and $V^n_{j,s(n)} \nsubseteq U\ (1\le j\le m_{s(n)})\} \in \Pi_V(\Lambda)$.

We assert that:

$$\{St(\langle V^n_{1,s(n)},\dots,V^n_{m_{s(n)},s(n)}\rangle,\mathcal{U}_n) : n\in\mathbb{N}\}$$

is an open cover of Λ.

Indeed, let $A \in \Lambda$. Since $\mathcal{J}$ is a $\pi_V(\Lambda)$-network of X and $A^c \in \Lambda^c$, there exist $(V^{n_0}_{1,s_0}, \ldots, V^{n_0}_{m_{s_0},s_0}) \in \mathcal{J}$ (for some $n_0 \in \mathbb{N}$, $s_0 \in S_{n_0}$) and $F \in [X]^{<\omega}$ such that for each $i \in \{1, \ldots, m_{s_0}\}$, $F \cap V^{n_0}_{i,s_0} \neq \emptyset$, $\bigcap_{i=1}^{m_s}(V^{n_0}_{i,s_0})^c \subseteq A^c$ and $F \cap A^c = \emptyset$. Hence, $A \subseteq \bigcup_{i=1}^{m_s} V^{n_0}_{i,s_0}$ and, for each $i \in \{1, \ldots, m_{s_0}\}$, $A \cap V^{n_0}_{i,s_0} \neq \emptyset$, that is, $A \in \langle V^{n_0}_{1,s_0}, \ldots, V^{n_0}_{m_{s_0},s_0}\rangle$.

Now, since $(V^{n_0}_{1,s_0}, \ldots, V^{n_0}_{m_{s_0},s_0}) \in \mathcal{J}$, there exists $U \in \Lambda^c$ that satisfies $\left[\bigcap_{i=1}^{m_s} V^c_{i,s}\right] \cup \left[\bigcap_{i=1}^{m_{s(n)}} V^c_{i,s(n)}\right] \subseteq U$, $V^n_{i,s} \nsubseteq U$ $(1 \leq i \leq m_s)$ and $V^n_{j,s(n)} \nsubseteq U$ $(1 \leq j \leq m_{s(n)})$. Then $U^c \in \langle V^{n_0}_{1,s}, \ldots, V^{n_0}_{m_s,s}\rangle \cap \langle V^{n_0}_{1,s(n_0)}, \ldots, V^{n_0}_{m_{s(n_0)},s(n_0)}\rangle$. Thus, this intersection is non empty and we get that:

$$A \in St(\langle V^{n_0}_{1,s(n_0)}, \ldots, V^{n_0}_{m_{s(n_0)},s(n_0)}\rangle, \mathcal{U}_{n_0}).$$

Hence, $\{St(\langle V^n_{1,s(n)}, \ldots, V^n_{m_{s(n)},s(n)}\rangle, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover of Λ. □

As an immediate consequence of Theorem 2.11, we obtain the following result.

Corollary 2.12. *Let X be a topological space and let Λ be one of the following hyperspaces: $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$. Then Λ is SR if and only if X satisfies $\mathbf{S}^*_{\Pi_V}(\Pi_V(\Lambda), \Pi_V(\Lambda))$.*

The last theorem of this paper concerns to c_V-covers of X, a notion due to Li (see [5]). First of all, we make a modification to this concept and prove two results related with c_V-covers. For $V_1, \ldots, V_m$ open subsets of X, we put

$$\langle V_1, \ldots, V_m\rangle_\Lambda = \langle V_1, \ldots, V_m\rangle \cap \Lambda$$

to denote relative open subsets of the subspace $\Lambda \subseteq CL(X)$.

Definition 2.13. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $c_V(\Lambda)$-cover of X if for any open subsets $V_1, \ldots, V_m$ of X with $\langle V_1, \ldots, V_m\rangle_\Lambda \neq \emptyset$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that for each $i \in \{1, \ldots, m\}$, $F \cap V_i \neq \emptyset$, $\bigcap_{i=1}^m V^c_i \subseteq U$ and $F \cap U = \emptyset$.

We denote by $\mathbb{C}_V(\Lambda)$ the family of $c_V(\Lambda)$-covers of X and by $\mathcal{D}$ the family of dense subsets of a space.

Lemma 2.14. *A family $\mathcal{U} \subseteq \Lambda^c$ is a $c_V(\Lambda)$-cover of X if and only if the family $\mathcal{U}^c$ is a dense subset of Λ.*

Proof. Suppose that $\mathcal{U}$ is a $c_V(\Lambda)$-cover of X. Let $\langle V_1, \ldots, V_m\rangle_\Lambda$ be a nonempty open subset of Λ. By hypothesis, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that, for each $i \in \{1, \ldots, m\}$, $F \cap V_i \neq \emptyset$, $\bigcap_{i=1}^m V^c_i \subseteq U$ and $F \cap U = \emptyset$. Let $A = U^c$. We have that $A \subseteq \bigcup_{i=1}^m V_i$ and for every $i \in \{1, \ldots, m\}$, $A \cap V_i \neq \emptyset$. So, $A \in \langle V_1, \ldots, V_m\rangle_\Lambda \cap \mathcal{U}^c$. Therefore, $\mathcal{U}^c$ is dense in Λ.

On the other hand, suppose that $\mathcal{U}^c$ is a dense subset of Λ. Let $V_1, \ldots, V_m$ be open subsets of X with $\langle V_1, \ldots, V_m\rangle_\Lambda \neq \emptyset$. By hypothesis, we obtain that $\langle V_1, \ldots, V_m\rangle_\Lambda \cap \mathcal{U}^c \neq \emptyset$. Fix $D \in \langle V_1, \ldots, V_m\rangle_\Lambda \cap \mathcal{U}^c$. It is clear that $D^c \in \mathcal{U}$. Moreover, since for each $i \in \{1, \ldots, m\}$, $D \cap V_i \neq \emptyset$, then we can consider, for each $i \in \{1, \ldots, m\}$, $x_i \in D \cap V_i$. Let $F = \{x_i : i \in \{i, \ldots, m\}\}$. Clearly, $F \in [X]^{<\omega}$ and for each $i \in \{1, \ldots, m\}$, $F \cap V_i \neq \emptyset$. Observe that $\bigcap_{i=1}^m V^c_i \subseteq D^c$ and $F \cap D^c = \emptyset$. So, $\mathcal{U}$ is a $c_V(\Lambda)$-cover of X. □

Theorem 2.15. *For a space X the following conditions are equivalent:*

(1) *Λ satisfies $\mathbf{S}_1(\mathcal{D}, \mathcal{D})$;*
(2) *X satisfies $\mathbf{S}_1(\mathbb{C}_V(\Lambda), \mathbb{C}_V(\Lambda))$.*

Proof. (1) $\Rightarrow$ (2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ a sequence of $c_V(\Lambda)$-covers of X. We put, for each $n \in \mathbb{N}$, $\mathcal{A}_n = \mathcal{U}_n^c$. From Lemma 2.14, we obtain that, for each $n \in \mathbb{N}$, $\mathcal{A}_n$ is a dense subset of Λ.

Then, applying (1) to the sequence $(\mathcal{A}_n : n \in \mathbb{N})$, we obtain, for each $n \in \mathbb{N}$, $A_n \in \mathcal{A}_n$ such that $\{A_n : n \in \mathbb{N}\}$ is a dense subset of Λ. Put, for each $n \in \mathbb{N}$, $U_n = A_n^c$. We have that $\{U_n : n \in \mathbb{N}\} \in \mathbb{C}_V(\Lambda)$. Indeed, let $V_1, \ldots, V_m$ be open subsets of X such that $\langle V_1, \ldots, V_m\rangle_\Lambda \neq \emptyset$. Since $\{A_n : n \in \mathbb{N}\}$ is a dense subset of Λ, it follows that $\langle V_1, \ldots, V_m\rangle_\Lambda \cap \{A_n : n \in \mathbb{N}\} \neq \emptyset$. Fix $A_p \in \langle V_1, \ldots, V_m\rangle_\Lambda \cap \{A_n : n \in \mathbb{N}\}$, for some $p \in \mathbb{N}$. Clearly, $A_p^c \in \{U_n : n \in \mathbb{N}\}$. Moreover, since $A_p \in \langle V_1, \ldots, V_m\rangle_\Lambda$, we can consider for each $i \in \{1, \ldots, m\}$, $x_i \in A_p \cap V_i$. Let $F = \{x_i : i \in \{1, \ldots, m\}\}$. Observe that $F \in [X]^{<\omega}$ and for each $i \in \{1, \ldots, m\}$, $F \cap V_i \neq \emptyset$. On the other hand, since $A_p \subseteq \bigcup_{i=1}^m V_i$, then $\bigcap_{i=1}^m V_i^c \subseteq A_p^c$. Finally, since $F \subseteq A_p$, we have that $F \cap A_p^c = \emptyset$. So, $\{A_n : n \in \mathbb{N}\}$ is $c_V(\Lambda)$-cover of X. Therefore, X satisfies $\mathbf{S}_1(\mathbb{C}_V(\Lambda), \mathbb{C}_V(\Lambda))$.

(2) $\Rightarrow$ (1) Let $(\mathcal{D}_n : n \in \mathbb{N})$ be a sequence of dense subsets of Λ. For each $n \in \mathbb{N}$, we put $\mathcal{U}_n = \mathcal{D}_n^c$. It follows from Lemma 2.14 that, for each $n \in \mathbb{N}$, $\mathcal{U}_n$ is a $c_V(\Lambda)$-cover of X.

Thus, applying (2) to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$, there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\}$ is a $c_V(\Lambda)$-cover of X. Put, for each $n \in \mathbb{N}$, $D_n = U_n^c$. Then, for each $n \in \mathbb{N}$, $D_n \in \mathcal{D}_n$. We claim that $\{D_n : n \in \mathbb{N}\}$ is a dense subset of Λ. Let $\langle V_1, \ldots, V_m\rangle_\Lambda$ be a nonempty open subset of Λ. Given that $\{U_n : n \in \mathbb{N}\}$ is a $c_V(\Lambda)$-cover of X, there exists $U_k \in \{U_n : n \in \mathbb{N}\}$, for some $k \in \mathbb{N}$, and there exists $F \in [X]^{<\omega}$ such that for each $i \in \{1, \ldots, m\}$, $F \cap V_i \neq \emptyset$, $\bigcap_{i=1}^m V_i^c \subseteq U_k$ and $F \cap U_k = \emptyset$. Let $D = U_k^c$. Clearly, $D \in \{D_n : n \in \mathbb{N}\}$ and $D \in \langle V_1, \ldots, V_m\rangle_\Lambda$. So, $\{D_n : n \in \mathbb{N}\}$ is a dense subset of Λ. Therefore, Λ satisfies $\mathbf{S}_1(\mathcal{D}, \mathcal{D})$. $\square$

As an immediate consequence of Theorem 2.15, we have the following result.

Corollary 2.16. *Let X be a topological space and let Λ be one of the following hyperspaces: $CL(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$. Then Λ has the $\mathbf{S}_1(\mathcal{D}, \mathcal{D})$ if and only if X satisfies $\mathbf{S}_1(\mathbb{C}_V(\Lambda), \mathbb{C}_V(\Lambda))$.*

Observe that when $\Lambda = CL(X)$ in Corollary 2.16, we obtain [5, Theorem 3.6].

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